5.3 Colouring edges

Clearly, every graph G satisfies $\chi'(G) \geqslant \Delta(G)$. For bipartite graphs, we have equality here:

Proposition 5.3.1. (König 1916) [5.4.5]
Every bipartite graph G satisfies $\chi'(G) = \Delta(G)$.

Proof. We apply induction on $\|G\|$. For $\|G\| = 0$ the assertion holds. (1.6.1)
Now assume that $\|G\| \geqslant 1$, and that the assertion holds for graphs with fewer edges. Let $\Delta := \Delta(G)$, pick an edge $xy \in G$, and choose a Δ-edge-colouring of $G - xy$ by the induction hypothesis. Let us refer to the edges coloured α as *α-edges*, etc. Δ, xy *α-edge*

In $G - xy$, each of x and y is incident with at most $\Delta - 1$ edges. Hence there are $\alpha, \beta \in \{1, \dots, \Delta\}$ such that x is not incident with an α-edge and y is not incident with a β-edge. If $\alpha = \beta$, we can colour the edge xy with this colour and are done; so we may assume that $\alpha \neq \beta$, and that x is incident with a β-edge. α, β

Let us extend this edge to a maximal walk W from x whose edges are coloured β and α alternately. Since no such walk contains a vertex twice (why not?), W exists and is a path. Moreover, W does not contain y: if it did, it would end in y on an α-edge (by the choice of β) and thus have even length, so $W + xy$ would be an odd cycle in G (cf. Proposition 1.6.1). We now recolour all the edges on W, swapping α with β. By the choice of α and the maximality of W, adjacent edges of $G - xy$ are still coloured differently. We have thus found a Δ-edge-colouring of $G - xy$ in which neither x nor y is incident with a β-edge. Colouring xy with β, we extend this colouring to a Δ-edge-colouring of G. □

Theorem 5.3.2. (Vizing 1964)
Every graph G satisfies

$$\Delta(G) \leqslant \chi'(G) \leqslant \Delta(G) + 1\,.$$

Proof. We prove the second inequality by induction on $\|G\|$. For $\|G\| = 0$ it is trivial. For the induction step let $G = (V, E)$ with $\Delta := \Delta(G) > 0$ be given, and assume that the assertion holds for graphs with fewer edges. Instead of '$(\Delta + 1)$-edge-colouring' let us just say 'colouring'. An edge coloured α will again be called an *α-edge*. V, E Δ *colouring* *α-edge*

For every edge $e \in G$ there exists a colouring of $G - e$, by the induction hypothesis. In such a colouring, the edges at a given vertex v use at most $d(v) \leqslant \Delta$ colours, so some colour $\beta \in \{1, \dots, \Delta + 1\}$ is *missing at v*. For any other colour α, there is a unique maximal walk (possibly trivial) starting at v, whose edges are coloured alternately α and β. This walk is a path; we call it the *α/β - path from v*. *missing* *α/β - path*

Suppose that G has no colouring. Then the following holds: