Theorem 10.1.1. (Dirac 1952)
Every graph with $n \geqslant 3$ vertices and minimum degree at least $n/2$ has a Hamilton cycle.

Proof. Let $G = (V, E)$ be a graph with $|G| = n \geqslant 3$ and $\delta(G) \geqslant n/2$. Then G is connected: otherwise, the degree of any vertex in the smallest component C of G would be less than $|C| \leqslant n/2$.

Let $P = x_0 \dots x_k$ be a longest path in G. By the maximality of P, all the neighbours of x_0 and all the neighbours of x_k lie on P. Hence at least $n/2$ of the vertices $x_0, \dots, x_{k-1}$ are adjacent to x_k, and at least $n/2$ of these same $k < n$ vertices x_i are such that $x_0 x_{i+1} \in E$. By the pigeon hole principle, there is a vertex x_i that has both properties, so we have $x_0 x_{i+1} \in E$ and $x_i x_k \in E$ for some $i < k$ (Fig. 10.1.1).

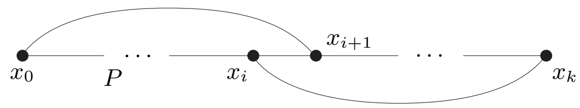

Fig. 10.1.1. Finding a Hamilton cycle in the proof Theorem 10.1.1

We claim that the cycle $C := x_0 x_{i+1} P x_k x_i P x_0$ is a Hamilton cycle of G. Indeed, since G is connected, C would otherwise have a neighbour in $G - C$, which could be combined with a spanning path of C into a path longer than P. □

Theorem 10.1.1 is best possible in that we cannot replace the bound of $n/2$ with $\lfloor n/2 \rfloor$: if n is odd and G is the union of two copies of $K^{\lceil n/2 \rceil}$ meeting in one vertex, then $\delta(G) = \lfloor n/2 \rfloor$ but $\kappa(G) = 1$, so G cannot have a Hamilton cycle. In other words, the high level of the bound of $\delta \geqslant n/2$ is needed to ensure, if nothing else, that G is 2-connected: a condition just as trivially necessary for hamiltonicity as a minimum degree of at least 2. It would seem, therefore, that prescribing some high (constant) value for κ rather than for δ stands a better chance of implying hamiltonicity. However, this is not so: although every large enough k-connected graph contains a cycle of length at least $2k$ (Ex. 16, Ch. 3), the graphs $K_{k,n}$ show that this is already best possible.

Slightly more generally, a graph G with a separating set S of k vertices such that $G - S$ has more than k components is clearly not hamiltonian. Could it be true that all non-hamiltonian graphs have such a separating set, one that leaves many components compared with its size? We shall address this question in a moment.

For now, just note that such graphs as above also have relatively large independent sets: pick one vertex from each component of $G - S$ to obtain one of order at least $k + 1$. Might we be able to force a Hamilton cycle by forbidding large independent sets?