CSE 301: Class Test 1 Answer Sheet

Question 1: A triple Tower of Hanoi contains $3n$ disks of n different sizes, three of each size. As usual, we are required to move only one disk at a time, without putting a larger disk over a smaller one. How many moves does it take to transfer a triple tower from one peg to another, if the ordering of the disks of equal size is not important? Follow the following steps:

(a) Describe your algorithm and from there, give an upper bound.

(b) Prove that your upper bound is also a lower bound.

(c) Then find the closed form of the recursive equation by any method you like.

Answer: [This question is from Exercise 1.11, 1.12 of [Concrete].]
(a) Algorithm [4 marks]:

1. Move recursively top $3(n-1)$ discs to the middle peg.
2. Then move the bottom three discs (which are of same size) by three moves to the destination peg.
3. Finally, recursively move the $3(n-1)$ discs from the middle peg to the destination peg.

Upper bound: Let A_n denote the number of moves required to move $3n$ discs to the destination peg. Then our first and last step require A_{n-1} moves each. The second sep requires 3 moves. So we get,

$$A_n \leq A_{n-1} + 3 + A_{n-1} \qquad \text{(I)}$$

(b) Lower bound [4 marks]: We will prove that no other algorithm can do better than the above equation. Let L_n means the best possible (i.e., minimum) number of moves required by any best algorithm. Any best algorithm must move at some point the three bottom (largest) discs to the target peg, and for that, it requires *at least* three moves. But when that algorithm does that (that means, moves that three bottom discs) it must preempt the top $3(n-1)$ discs to a different peg, which is equivalent to moving a triple Tower of Hanoi of $3(n-1)$ discs *at least* once. So, recursively, it will require *at least* L_{n-1} moves. Again, after moving the largest three discs, the algorithm will require to move the triple Tower of Hanoi of $3(n-1)$ discs to the target peg, which again requires *at least* L_{n-1} moves recursively. So, in summary, the minimum number of moves required by the best algorithm is,

$$L_n \geq L_{n-1} + 3 + L_{n-1} \qquad \text{(II)}$$

From (I) and (II), we can say that the worst-case number of moves required by our algorithm, which is $A_{n-1} + 3 + A_{n-1}$ from (I), is exactly same as the minimum number of moves required by any best algorithm, which is $L_{n-1} + 3 + L_{n-1}$ from (II), if $A_0 = L_0$. For 0 disc, since the maximum number of move required by our algorithm is 0, and since the minimum number of moves required by any best algorithm is also 0, we have $A_0 = L_0 = 0$. So, $A_{n-1} + 3 + A_{n-1} = L_{n-1} + 3 + L_{n-1}$. It gives that $A_n = L_n$ is the best possible moves, which is achieved by our algorithm. Therefore, the recursive formula we get is,

$$A_0 = 0$$

$$A_n = A_{n-1} + 3 + A_{n-1}$$

(c) [2 marks] This part is a high school math problem and I do not want to provide the solution. The final answer is $A_n = 3 \cdot (2^n - 1)$. Note that it is three times the solution of single Tower of Hanoi probelm. ■

Question 2:

$$\sum_{1 \le j \le k \le n} a j a_k = \frac{1}{2}\left(\left(\sum_{k=1}^{n} a_k\right)^2 + \sum_{k=1}^{n} a_k^2\right)$$

Answer: See [Concrete] Page 37 for the answer. ■

Question 3: Suppose there are $3n$ people in a circle; the first n are "good guys", the middle n are "mixed of good and bad guys" and the last n are "bad guys". Find an integer m (in terms of n) such that if we go around the circle executing every m-th person, then all bad guys are to go first, all the middle n guys are next to go, and finally all the good guys are to go. Justify your answer.

Answer: Let me first clarify the question. There might be some confusion in the question: what does it means by "all bad guys are to go first"? Is it the last n bad guys only? Or all bad guys includig those in the middle segment of n persons? Moreover, for later case how do I know what is the number of bad guys in the middle segmnent and what are their positions? But the confusion should go away after reading the next sentence in the question, which says that "all the middle n guys are next to go". So the question actualy means that the last n guys are to go first, then the middle n guys, and finally, the first n guys are to go.

Now come to the solution [3 marks]. This question is a modified version of the exercise 1.21 of [Concrete]. We can assign m as the lowest common multiple (lcm) of $3n, (3n-1), \ldots, (n+1)$.

Justifcation [2 marks]: Since m is divisible by each of $3n, (3n-1), \ldots, (n+2)$, and $(n+1)$, at the very fist time the $3n$-th person will go, then the $(3n-1)$-th person, and so on up to $(n+1)$-th person in the order. Then the remaining n person will go. ■